1. In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

Find $\int_0^{\frac{\pi}{6}} x\cos 3x\ dx$ **giving your answer in simplest form.** **(5)**

Sea $F(x) = \int x\cos 3x\ dx$. Integrando por partes

$$\left.\begin{array}{c} x = u\ ,\ dx = du \\ \cos 3x\ dx = dv\ ,\ v = \int 3x\ dx = \frac{1}{3}\sin 3x \end{array}\right\} \Rightarrow$$

$$F(x) = \int x\cos 3x\ dx = \frac{x\cdot\sin 3x}{3} - \int \frac{1}{3}\sin 3x\ dx = \frac{x\cdot\sin 3x}{3} + \frac{1}{9}\cos 3x + C$$

Por lo tanto $\int_0^{\frac{\pi}{6}} x\cos 3x\ dx = F\left(\frac{\pi}{6}\right) - F(0) = \frac{\frac{\pi}{6}\cdot\sin\frac{\pi}{2}}{3} + \frac{\cos\frac{\pi}{2}}{9} + C - \frac{0\cdot\sin 0}{3} + \frac{\cos 0}{9} - C = \frac{\pi}{18} - \frac{1}{9}$

2. With respect to a fixed origin, O, the point A has position vector $\overrightarrow{OA} = \begin{pmatrix} 7 \\ 2 \\ -5 \end{pmatrix}$**. Given that** $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$

(a) find the coordinates of the point B. **(2)**

Dado que $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \Rightarrow \overrightarrow{OB} = \begin{pmatrix} 7 \\ 2 \\ -5 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix}$

The point C has position vector $\overrightarrow{OC} = \begin{pmatrix} a \\ 5 \\ -1 \end{pmatrix}$ **where a is a constant. Given that** $\overrightarrow{OC}$ **is perpendicular to** $\overrightarrow{BC}$

(b) find the possible values of a. **(4)**

Hallamos primero el vector $\overrightarrow{BC}$. Dado que $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC} \Rightarrow \overrightarrow{BC} = \begin{pmatrix} a \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} a-5 \\ -1 \\ 1 \end{pmatrix}$. Dado que los vectores $\overrightarrow{OC}$ y $\overrightarrow{BC}$ son perpendiculares, su producto es calar debe ser cero.

$$\overrightarrow{OC}\cdot\overrightarrow{BC} = 0 \Rightarrow \begin{pmatrix} a & 5 & -1 \end{pmatrix}\cdot\begin{pmatrix} a-5 \\ -1 \\ 1 \end{pmatrix} = 0 \Rightarrow a^2 - 5a - 5 - 1 = 0 \Rightarrow a^2 - 5a - 6 = 0 \Rightarrow a = \begin{cases} -1 \\ 6 \end{cases}$$

3. The curve C is defined by the equation $8x^3 - 3y^2 + 2xy = 9$**. Find an equation of the normal to C at the point** $(2, 5)$**, giving your answer in the form** $ax + by + c = 0$**, where a, b and c are integers.** **(7)**

La derivada $\frac{dy}{dx}$ en $(2, 5)$ nos proporcionará la pendiente m de la recta tangente en dicho punto. La pendiente m' de la recta normal la hallaremos imponiendo la condición $m\cdot m' = -1$.

Haciendo la derivada implícita de C

$$24x^2 - 6y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = \frac{24x^2 + 2y}{6y - 2x} \;\Rightarrow\; \left.\frac{dy}{dx}\right|_{(2,5)} = \frac{24 \cdot 2^2 + 2 \cdot 5}{6 \cdot 5 - 2 \cdot 2} = \frac{53}{13}$$

Por tanto la pendiente de la recta normal a C en el punto $(2,5)$ es $-\frac{13}{53}$ y la ecuación de dicha recta queda

$$y - 5 = -\frac{13}{53}(x - 2) \;\Rightarrow\; 13x + 53y - 291 = 0$$

4.

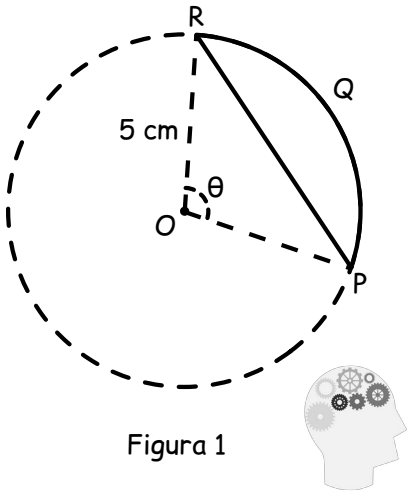

Figura 1

Figure 1 shows a sketch of a segment PQRP of a circle with centre O and radius 5 cm. Given that

- **angle POR is θ radians**
- **θ is increasing, from 0 to π, at a constant rate of 0.1 radians per second**
- **the area of the segment PQRP is A cm²**

(a) show that $\frac{dA}{d\theta} = K(1 - \cos\theta)$ where K is a constant to be found. **(2)**

El área del segmento PQRP será el del sector circular OPQRO $\left(\frac{1}{2}\theta \cdot r^2\right)$menos el del triángulo OPRO $\left(\frac{1}{2}ab \cdot \sin C\right)$

$$A = \frac{1}{2}\theta \cdot 5^2 - \frac{1}{2}5^2 \cdot \sin\theta = \frac{25}{2}(1 - \sin\theta)$$

Y derivando respecto a θ

$$\frac{dA}{d\theta} = \frac{25}{2}(1 - \cos\theta)$$

(b) Find, in cm² s⁻¹, the rate of increase of the area of the segment when θ = π/3 **(4)**

La tasa de variación del área del segmento PQRP viene dada por su derivada con respecto al tiempo que puede escribirse como $\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt}$ y dado que la tasa de variación del ángulo θ es constante nos queda

$$\left.\frac{dA}{dt}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{dA}{d\theta}\right|_{\theta=\frac{\pi}{3}} \frac{d\theta}{dt} = \frac{25}{2}\left(1 - \cos\frac{\pi}{3}\right) \cdot 0.1 = \frac{25}{4} \cdot 0.1 = \frac{5}{8}\ cm^2 \cdot s^{-1}$$

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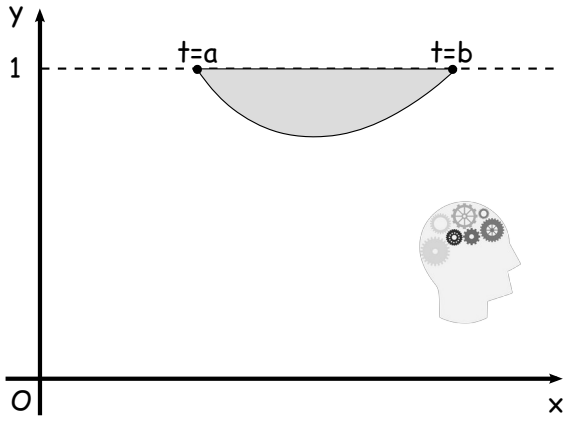

Figura 2

Figure 2 shows a sketch of the curve defined by the parametric equations $x = t^2 + 2t\ ,\ y = \dfrac{2}{t(3-t)}\ ,\ a \le y \le b$ **where a and b are constants.**

The ends of the curve lie on the line with equation y = 1

(a) Find the value of a and the value of b **(2)**

$$1 = \frac{2}{t\,(3-t)} \ \Rightarrow\ t\,(3-t) = 2 \ \Rightarrow\ \ -t^2 + 3t - 2 = 0 \ \Rightarrow\ t = \begin{cases} 1 \ \Rightarrow\ x = 3 \\ 2 \ \Rightarrow\ x = 8 \end{cases}$$

The region R, shown shaded in Figure 2, is bounded by the curve and the line with equation y = 1

(b) Show that the area of region R is given by $M - k\displaystyle\int_a^b \frac{t+1}{t\,(3-t)}dt$ **where M and k are constants to be found.** **(5)**

El área sombreada es el área del rectángulo de limitado por las rectas $y = 1,\ \ x = 3$ y $x = 8$, menos el área que queda limitada por la curva, el eje de abscisas y las rectas $x = 3$ y $x = 8$. Por tanto

$$A = (8-3)\cdot 1 - \int_3^8 y(x)\ dx = 5 - \int_1^2 y(t)\frac{dx}{dt}\ dt = 5 - \int_1^2 \frac{2}{t\,(3-t)}\,(2t+2)\ \ dt = 5 - 4\int_1^2 \frac{t+1}{t\,(3-t)}\ dt$$

(c) (i) Write $\dfrac{t+1}{t\,(3-t)}$ **in partial fractions.**

$$\frac{t+1}{t\,(3-t)} = \frac{A}{t} + \frac{B}{3-t} \ \Rightarrow\ t+1 = A\cdot(3-t) + B\cdot t \ \ \Rightarrow\ \begin{cases} t = 0 \ \Rightarrow\ 1 = 3A\ ,\ A = \frac{1}{3} \\ \\ t = 3 \ \Rightarrow\ 4 = 3B\ ,\ B = \frac{4}{3} \end{cases}$$

(ii) Use algebraic integration to find the exact area of R, giving your answer in simplest form. **(6)**

$$A = 5 - 4\int_1^2 \frac{t+1}{t\,(3-t)}\ dt = 5 - 4\int_1^2 \left(\frac{\frac{1}{3}}{t} + \frac{\frac{4}{3}}{3-t}\right)\ dt = 5 - 4\left[\frac{1}{3}\ln t - \frac{4}{3}\ln(3-t)\right]\Bigg|_1^2 =$$

$$= 5 - 4\left(\frac{1}{3}\ln 2 + \frac{4}{3}\ln 2\right) = 5 - \frac{20}{3}\ln 2 \ u^2$$

6. With respect to a fixed origin O, the line l_1 is given by the equation $\vec{r} = \hat{i} + 2\hat{j} + 5\hat{k} + \lambda\left(8\hat{i} - \hat{j} + 4\hat{k}\right)$ where λ is a scalar parameter. The point A lies on l_1 Given that $\left|\overrightarrow{OA}\right| = 5\sqrt{10}$

(a) show that at A the parameter λ satisfies $81\lambda^2 + 52\lambda - 220 = 0$ **(3)**

El vector de posición de un punto genérico X de la recta l_1 es $\overrightarrow{OX} = (8\lambda + 1)\,\hat{i} + (2 - \lambda)\,\hat{j} + (5 + 4\lambda)\,\hat{k}$. Buscamos aquel que tiene por módulo $5\sqrt{10}$, por lo tanto

$$\left|\overrightarrow{OX}\right| = \sqrt{(8\lambda+1)^2 + (2-\lambda)^2 + (5+4\lambda)^2} = 5\sqrt{10} \ \Rightarrow \ 64\lambda^2 + 16\lambda + 1 + \lambda^2 - 4\lambda + 4 + 25 + 40\lambda + 16\lambda^2 = 250 \ \Rightarrow$$

$$\Rightarrow \ 81\lambda^2 + 52\lambda - 220 = 0$$

Hence
(b) (i) show that one possible position vector for A is $-15\hat{i} + 4\hat{j} - 3\hat{k}$

Resolviendo la ecuación cuadrática $81\lambda^2 + 52\lambda - 220 = 0 \ \Rightarrow \ \lambda = \begin{cases} \frac{110}{81} \\ -2 \end{cases}$

Para $\lambda = -2$, el vector de posición de X resulta

$$(8(-2) + 1)\,\hat{i} + (2 - (-2))\,\hat{j} + (5 + 4(-2))\,\hat{k} = -15\hat{i} + 4\hat{j} - 3\hat{k}$$

(ii) find the other possible position vector for A. **(3)**

Para $\lambda = \frac{110}{81}$, el vector de posición de X resulta

$$\left(8\left(\frac{110}{81}\right) + 1\right)\hat{i} + \left(2 - \left(\frac{110}{81}\right)\right)\hat{j} + \left(5 + 4\left(\frac{110}{81}\right)\right)\hat{k} = \frac{961}{81}\hat{i} + \frac{52}{81}\hat{j} + \frac{845}{81}\hat{k}$$

The line l_2 is parallel to l_1 and passes through O. Given that

- $\overrightarrow{OA} = -15\hat{i} + 4\hat{j} - 3\hat{k}$
- **point B lies on l_2 where $\left|\overrightarrow{OB}\right| = 4\sqrt{10}$**

(c) find the area of triangle OAB, giving your answer to one decimal place. **(4)**

El área del triángulo vendrá dado por $A = \frac{1}{2}\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right| \cdot \sin\theta$, siendo θ el ángulo que forman los vectores $\overrightarrow{OA}$ y $\overrightarrow{OB}$. Para hallarlo podríamos emplear su producto escalar, pero no conocemos el vector $\overrightarrow{OB}$. La ecuación de la recta l_2 pasa por el origen y su vector director es el mismo que el de l_1. Por tanto su ecuación viene dada por $\vec{r} = \tau\left(8\hat{i} - \hat{j} + 4\hat{k}\right)$. Así $\overrightarrow{OB}$ es un vector paralelo al vector director $\vec{w}$ de la recta l_2 y por tanto el ángulo que forman $\overrightarrow{OA}$ y $\overrightarrow{OB}$ es el mismo que el que forman $\overrightarrow{OA}$ y $\vec{w}$. Por lo tanto

$$\cos\theta = \frac{\overrightarrow{OA}\cdot\vec{w}}{\left|\overrightarrow{OA}\right|\,|\vec{w}|} = \frac{(-15\ 4\ -3)\begin{pmatrix}8\\-1\\4\end{pmatrix}}{5\sqrt{10}\cdot\sqrt{69}} = \frac{-136}{45\sqrt{10}} \Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{\frac{877}{10125}}$$

$$A = \frac{1}{2}\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cdot\sin\theta = \frac{1}{2}5\sqrt{10}\cdot 4\sqrt{10}\cdot\sqrt{\frac{877}{10125}} = \frac{20}{9}\sqrt{\frac{877}{5}} = 29.4\ u^2$$

7. The current, x amps, at time t seconds after a switch is closed in a particular electric circuit is modelled by the equation $\frac{dx}{dt} = k - 3x$ where k is a constant. Initially there is zero current in the circuit.

(a) Solve the differential equation to find an equation, in terms of k, for the current in the circuit at time t seconds. Give your answer in the form x = f(t). **(6)**

$$\frac{dx}{k-3x} = dt \Rightarrow \int_0^x \frac{dx}{k-3x}\,dx = \int_0^t dt \Rightarrow -\frac{1}{3}\ln(k-3x)\Big|_0^x = t\Big|_0^t \Rightarrow \frac{1}{3}\ln k - \frac{1}{3}\ln(k-3x) = t \Rightarrow$$

$$\Rightarrow \frac{1}{3}\ln\left(\frac{k}{k-3x}\right) = t \Rightarrow \ln\left(\frac{k}{k-3x}\right) = 3t \Rightarrow \frac{k}{k-3x} = e^{3t} \Rightarrow \frac{k-3x}{k} = e^{-3t} \Rightarrow x = \frac{k}{3}\left(1-e^{-3t}\right)$$

Given that in the long term the current in the circuit approaches 7 amps,
(b) find the value of k. **(2)**

Cuando $t \longrightarrow \infty \Rightarrow e^{-3t} \longrightarrow 0 \Rightarrow x \longrightarrow \frac{k}{3} = 7 \Longrightarrow k = 21$

(c) Hence find the time in seconds it takes for the current to reach 5 amps, giving your answer to 2 significant figures. **(3)**

$$5 = 7\left(1-e^{-3t}\right) \Rightarrow \frac{5}{7} - 1 = -e^{-3t} \Rightarrow \frac{2}{7} = e^{-3t} \Rightarrow \ln\left(\frac{2}{7}\right) = -3t \Rightarrow t = \frac{\ln\left(\frac{2}{7}\right)}{-3} = 0.42\ s$$

8. $$f(x) = (8-3x)^{\frac{4}{3}}\ ,\ 0 < x < \frac{8}{3}$$

(a) Show that the binomial expansion of f(x) in ascending powers of x up to and including the term in x^3 is $A - 8x + \frac{x^2}{2} + Bx^3 + \ldots$ where A and B are constants to be found. **(4)**

$$(8-3x)^{\frac{4}{3}} = 8^{\frac{4}{3}}\cdot\left(1-\frac{3}{8}x\right)^{\frac{4}{3}} = 16\cdot\left(1-\frac{4}{3}\cdot\frac{3x}{8}+\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{1}{3}\left(\frac{3x}{8}\right)^2-\frac{1}{6}\cdot\frac{4}{3}\cdot\frac{1}{3}\cdot\left(\frac{-2}{3}\right)\cdot\left(\frac{3x}{8}\right)^3+\ldots\right) \approx$$

$$\approx 16\left(1-\frac{1}{2}x+\frac{1}{32}x^2+\frac{1}{384}x^3\right) = 16-8x+\frac{x^2}{2}+\frac{x^3}{24}$$

(b) Use proof by contradiction to prove that the curve with equation $y = 8 + 8x - \frac{15}{2}x^2$ does not intersect the curve with equation $y = A - 8x + \frac{x^2}{2} + Bx^3 + \ldots\ ,\ 0 < x < \frac{8}{3}$ where A and B are the constants found in part (a). (Solutions relying on calculator technology are not acceptable.) (4)

Si ambas curvas se cruzan entonces

$$8 + 8x - \frac{15}{2}x^2 = 16 - 8x + \frac{x^2}{2} + \frac{x^3}{24} \ \Rightarrow\ 8 - 16x + 8x^2 + \frac{x^3}{24} = 0 \ \Rightarrow\ 8(x-1)^2 + \frac{x^3}{24} = 0$$

Lo cual no tiene solución, ya que $8(x-1)^2 \geq 0$ y como $0 < x < \frac{8}{3} \ \Rightarrow\ \frac{x^3}{24} > 0 \ \Rightarrow\ 8(x-1)^2 + \frac{x^3}{24} > 0$ y por tanto las curvas no se cruzan.

9.

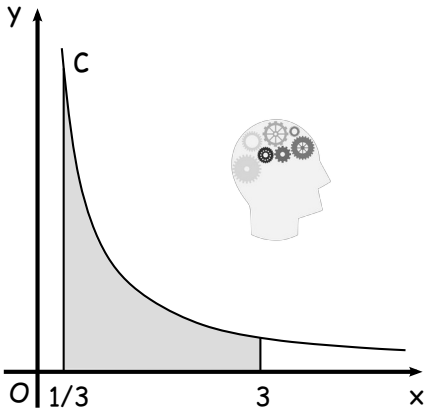

Figura 3

The curve C, shown in Figure 3, has equation $y = \frac{x^{\frac{-1}{4}}}{\sqrt{1+x}\arctan\sqrt{x}}$ .The region R, shown shaded in Figure 3, is bounded by C, the line with equation x = 3, the x-axis and the line with equation x = ⅓ . The region R is rotated through 360° about the x-axis to form a solid.

Using the substitution $\tan u = \sqrt{x}$

(a) show that the volume V of the solid formed is given by $k\int_a^b \frac{1}{u^2}\,du$ where k, a and b are constants to be found. (6)

$$V = \pi\int_{\frac{1}{3}}^{3} y^2\,dx = \pi\int_{\frac{1}{3}}^{3} \frac{x^{-\frac{1}{2}}}{(1+x)\arctan^2\sqrt{x}}\,dx$$

Haciendo el cambio de variable propuesto

$$\tan u = \sqrt{x} \ \Rightarrow\ u = \arctan\sqrt{x} \ \Rightarrow \begin{cases} du = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}\,dx \\ x = \frac{1}{3} \ \Rightarrow\ u = \arctan\sqrt{\frac{1}{3}} = \frac{\pi}{6} \\ x = 3 \ \Rightarrow\ u = \arctan\sqrt{3} = \frac{\pi}{3} \end{cases}$$

Por lo tanto

$$V = \pi \int_{\frac{1}{3}}^{3} \frac{1}{\arctan^2 \sqrt{x}} \cdot \frac{1}{1+x} \cdot \frac{1}{\sqrt{x}}\, dx = \pi \cdot 2 \int_{\frac{1}{3}}^{3} \frac{1}{\arctan^2 \sqrt{x}} \cdot \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}\, dx = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{u^2}\, du$$

(b) Hence, using algebraic integration, find the value of V in simplest form. **(3)**

$$V = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{u^2}\, du = -2\pi \frac{1}{u}\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = 2\pi \left(\frac{6}{\pi} - \frac{3}{\pi}\right) = 6\ u^3$$